International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 4, pp: 324-331 www.ijaem.net ISSN: 2395-5252

# Solution of Fourth Order Differential Equations Using Daubechies Wavelets

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Date of Submission: 1-08-2020 Date of Acceptance: 15-08-2020

**ABSTRACT:** In this paper, we obtain some special types of functions by integrating Daubechies wavelets which are differentiable and compactly supported. The resulting functions are used as Galerkin basis functions for numerical solution of differential equations. Theoretical and numerical results are obtained for elliptic problems of fourth order with various types of boundary conditions. Optimal error estimates are also obtained. Comparison of solutions with simple finite difference method suggests that for this types of problems, the present method will provide a better alternative to other classical methods. The methodology can be generalized to multidimensional problems.

**Keywords:** Wavelet; Numerical Solution; Boundary Value Problem; Galerkin Method; Finite Difference Method

## I. INTRODUCTION

In this paper, we study numerical solution of differential equations (DE) by using wavelets. Wavelets in our consideration are compactly supported Daubechies wavelets [5] which are differentiable. Since these functions combine orthogonality with localization and scaling properties, it is a very attractive idea to apply these functions to numerical solution of DE problems.

To discretize a DE problem by Wavelet-Galerkin method, the Galerkin basis is constructed from orthonormal bases of compactly supported wavelets which can be done in a number of ways. If wavelets are used in a direct way in such construction, due to lack of regularity, "low order" wavelets cannot be used and higher order wavelets result in tedious computations. Also, in this approach, setting of the boundary conditions is somewhat difficult. We can get rid of the difficulties, if we use a different approach where Galerkin basis functions are constructed by obtaining integrals of Daubechies functions in such away that the resulting functions satisfy various types of boundary conditions. Also, in this approach, one can pursue the solution process in higher dimension without computing the connection coefficients. However, in this approach, the resulting bases will not satisfy the multiresolution properties of wavelets which are satisfied by the bases in direct approach. In the present paper, we construct Galerkin basis functions for fourth order boundary value problems in any arbitrary domain (a,b) as follows:

(i) Shift the support of the wavelet from (0, 2N-1) to (a, b), where N is the order (or genus) of the wavelet.

(ii) Take the restrictions of dilates of translated scaling functions to (a, b).

(iii) obtain integrals of the functions taken in (ii) in (a,b) in a such a way that they satisfy the homogeneous form of the essential boundary conditions of the problem.

Fourth order problems can be solved by variational methods in different formulations such as (i) conventional formulation, (ii) Lagrange multiplier formulation, (iii) penalty function formulation, and (iv) mixed formulation. The details can be seen in Reddy [13]. The mixed formulation involves rewriting the given fourth order equation as a pair of second order equations by introducing secondary dependent variables. This decomposition of a higher order equation into a pair of lower order equations enables one to seek approximation in lower order (Sobolev) spaces. However, in this paper, we use the conventional formulation.

We compute numerical results and examine the convergence rates which are found to be better in comparison to finite difference solutions. The method can be generalized simply for simple (e, g, rectangular in two dimensions) domains in higher dimensions. In case of complex geometric domains, the wavelet method can be applied by combining with fictitious domain methods as in Wells and Zhou [17] or in Glowinski et al [8]. All the computations in this paper are done in MATLAB 6.1. The rest of the paper is organized as follows:

Section 2: Wavelet Integrals and Their Approximation Properties
Section 3: The Wavelet-Galerkin Method
Section 4: Numerical Results
Section 5: Conclusion

## II. WAVELET INTEGRALS AND THEIR APPROXIMATION PROPERTIES

### 2.1 Basic Properties of Daubechies Wavelets

Here we briefly recall the basic properties of Daubechies compactly supported wavelets. For details, we refer [5, 14].

For a positive integer N, consider two functions $\phi, \psi \in L^2(\mathbf{R})$ defined by

$$\phi(x) = \sum_k a_k \phi(2x-k), \quad \psi(x) = \sum_k b_k \phi(2x-k), \tag{2.1}$$

where $S = 2N-1$ and $\{a_k\}_{k\in Z}$ and $\{b_k\}_{k\in Z}$ are two sequences such that $a_k = b_k = 0$ for $k \notin \{0,1,\ldots S\}$ and satisfying some specific properties [5]. The functions $\phi$ and $\varphi$ are called $dbN$ scaling function and $dbN$ wavelet function respectively which are compa-ctly supported with supp($\phi$)=supp($\psi$)= $[0, S]$. These functions are available in wavelet toolbox of MATLAB 6 for $1 \le N \le 45$. They satisfy the following properties:

$$\begin{cases} \int_R \phi(x)dx = 1, \\ \sum_k \phi(x-k) = 1, \\ \int_R x^m \psi(x)dx = 0, \quad 0 \le m < N. \end{cases} \tag{2.2}$$

The translations and dilations of $\phi$ and $\psi$ are defined as

$$\phi_{n,k}(x) = 2^{n/2}\phi(2^n x - k), \quad \psi_{n,k}(x) = 2^{n/2}\psi(2^n x - k) \tag{2.3}$$

for $n, k \in Z$.

Now we define

$$V_n = L^2\text{-closure(span }\{\phi_{n,k} : k \in Z\}). \tag{2.4}$$

If $P_n$ be the orthogonal projection of $L^2(R)$ onto $V_n$, then we have

$$L^2(R) = \bigcup_n V_n, \tag{2.5}$$

in the sense that

$$P_n f \to f \quad \text{as} \quad n \to \infty. \tag{2.6}$$

### 2.2 Approximation of Function Spaces Using Wavelet Integrals

In this subsection, integrals of Daubechies scaling functions (the wavelet integrals) are used to approximate different function spaces (Sobolev spaces) useful for numerical solution of DE problems and we construct finite dimensional subspaces of these function spaces.

We recall that for an open interval (a, b) and for an integer $m \geq 1$, the space

$$H^m(a,b) = \{u \in H^{m-1}(a,b) : u' \in H^{m-1}(a,b)\} \tag{2.7}$$

is called the Sobolev space of order m, which is a Hilbert space with inner product $<.,.>_m$ given by

$$<u,v>_m = <u,v>_0 + <u',v'>_{m-1} \tag{2.8}$$

where $<u,v>_0 = \int_a^b uv\,dx,$ is the inner product on $L^2(a,b).$

Here we define the following subspaces of $H^m(a,b)$ which will be in use for the solution of DE problems in this paper.

$$H_0^m(a,b) = \{u \in H^m(a,b) : u(a) = u(b) = 0\},$$

$$H_-^m(a,b) = \{u \in H^m(a,b) : u(a) = u'(a) = 0\}.$$

Now the following statement can be implied by Theorem 1.1 in [7] which provides the basis of our approximation problem.

If $\phi$ be the $dbN$ scaling function, then there exists an integer m, $0 \leq m < N,$ such that the Sobolev space $H^m(a,b)$ can be approximated by the restrictions of translates and dilates of $\phi$ to $(a,b)$ and hence $H^{m+1}(a,b)$ can be approximated by their integrals.

We shift the support of $\phi$ from [0,S] to [a, b] by using the transformation

$$y = \frac{b-a}{S}x + a \tag{2.9}$$

Let

$$I_n = \{k \in Z : \text{supp}(\phi_{n,k}) \cap (a,b) \neq \phi\} \tag{2.10}$$

Considering $V_n$ as defined in (2.4), we define $V_n(a,b)$ to be the set of restrictions of all functions in $V_n$ to $(a,b).$ In fact, we take

$$V_n(a,b) = \text{span } \{\phi_{n,k}|_{(a,b)} : k \in I_n\} \tag{2.11}$$

The space $V_n(a,b)$, $n \geq 0,$ is a finite dimensional closed subspace of $H^m(a,b)$ and by Proposition 4.2 in [18], a basis of $V_n(a,b)$ can be taken as

$$\{\phi_{n,k} \in V_n(a,b) : 1 - S \leq k \leq 2^n S - 1\} \tag{2.12}$$

For $\phi_{n,k} \in V_n(a,b),$ we define the spaces $S_n^0(a,b)$ and $S_n^-(a,b)$ as

(i) $S_n^0(a,b) = \text{span}\{\Phi_{n,k} : k \in I_n\},$ where

$$\Phi_{n,k}(x) = \int_a^x \phi_{n,k}(s)ds - \frac{x-a}{b-a}\int_a^b \phi_{n,k}(x)dx \tag{2.13}$$

and

(ii) $S_n^-(a,b) = \text{span}\{\Phi_{n,k} : k \in I_n\},$ where

International Journal of Advances in Engineering and Management (IJAEM)
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$$\Phi_{n,k}(x) = \int_a^x \phi_{n,k}(s)ds - (x-a)\phi_{n,k}(a) \qquad (2.14)$$

Since $V_n(a,b)$ is a finite dimensional subspace of $H^m(a,b)$, therefore the spaces $S_n^0(a,b)$ and $S_n^-(a,b)$ are finite dimensional subspaces of $H_0^{m+1}(a,b)$ and $H_-^{m+1}(a,b)$ respectively whose bases can be taken as

$$\{\Phi_{n,k} \in S_n^0(a,b) : 2-S \le k \le 2^n S - 1\},$$
$$\{\Phi_{n,k} \in S_n^-(a,b) : 2-S \le k \le 2^n S - 1\}$$

respectively.

**Remark 2.1:** $k = 1-S$ is excluded in the above bases due to the fact that the set $\{\Phi_{n,k} : 1-S \le k \le 2^n S - 1\}$ is linearly dependent and the set $\{\Phi_{n,k} : 2-S \le k \le 2^n S - 1\}$ is linearly independent.

Figure 2.1 shows the pictures of the basis functions for $S_n^0(0,1)$ and $S_n^-(0,1)$ and their derivatives for $N = 3$ and $n = 1$.

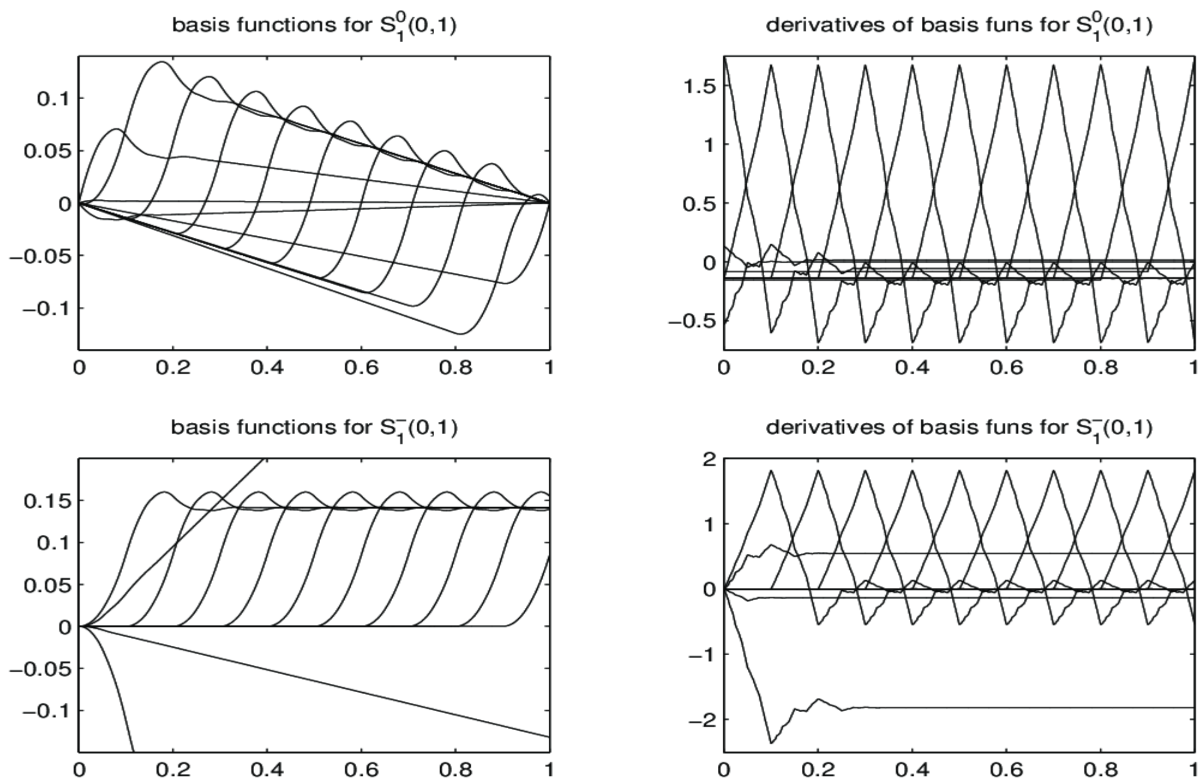


Figure 2.1: Basis funs and their derivatives for $S_1^0(0, 1)$ and $S_1^-(0, 1)$ for N=3

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## III. THE WAVELET-GALERKIN METHOD

In this section, we discuss the methodology for Wavelet-Galerkin solution of linear fourth order boundary value problems using the basis functions obtained in the last section. Two types of boundary conditions corresponding to the approximating spaces $S_n^0(a,b)$ and $S_n^-(a,b)$ are considered. Using numerical experiments in the next section, we shall show how the present algorithm gives fast solutions to DE problems. We shall also compare the convergent rates with finite difference method (FDM) which shows that the present algorithm is superior over FDM.

### 3.1 Formulation of the Problems

The problems to be discussed in this paper are of the form:

$$(\alpha(x)u'')'' + \beta(x)u = f(x) \tag{3.1}$$

with the boundary conditions:

$$\left.\begin{aligned} &(i)\ u(a) = c_1, && u(b) = c_2, \\ &(ii)\ (\alpha u'')(a) = d_{1,} && (\alpha u'')(b) = d_2 \end{aligned}\right\} \tag{3.2}$$

and

$$\left.\begin{aligned} &(i)\ u(a) = c_{1,} && u'(a) = c_2, \\ &(ii)\ (\alpha u'')(b) = d_1, && (\alpha u'')'(b) = d_2 \end{aligned}\right\} \tag{3.3}$$

We assume here that $f \in L^2(a,b)$ and the coefficients $\alpha$ and $\beta$ are differentiable in (a,b).

To solve a DE problem of the above type by Galerkin method, the problem is first converted into a linear variational (weak) problem:

$$\left.\begin{aligned} &\text{Find} \quad u \in H \quad \text{such that} \\ &A(u,v) = F(v), \quad \text{for all } v \in H \end{aligned}\right\} \tag{3.4}$$

where $H$ is a Hilbert space, $A(.,.)$ a bilinear form on $H$ and $F(.)$ a linear functional on $H$. By Lax-Milgram Lemma [4], problem (3.4) posseses a unique solution, if $A(.,.)$ is continuous and $H$ -elliptic and $F(.)$ is continuous. Then we find a family $\{H_n\}$ of finite dimensional closed subspaces of $H$ and problem (3.4) can be approximated on $\{H_n\}$ as

$$\left.\begin{aligned} &\text{Find} \quad u_n \in H_n \quad \text{such that} \\ &A(u_n, v_n) = F(v_n), \quad \text{for all } v_n \in H_n \end{aligned}\right\} \tag{3.5}$$

Now, the above two problems have the variational form:

$$\left.\begin{aligned} &u = u_0 + w, \quad w \in H, \\ &A(w,v) = F(v), \quad \textit{for all } v \in H \end{aligned}\right\} \tag{3.6}$$

where $A(.,.)$ is defined by

$$A(u,v) = \int_a^b (\alpha u''v'' + \beta uv)dx \tag{3.7}$$

and,

(i) for problem (3.1)-(3.2), $H = H_0^2(a,b)$, $u_0 = \dfrac{bc_1 - ac_2}{b-a} + \dfrac{c_2 - c_1}{b-a}x$, and

$$F(v)=\int_a^b fvdx+d_2v'(b)-d_1v'(a)-A(u_0,v) \qquad (3.8)$$

and, (ii) for problem (3.1)-(3.3), $H=H_-^2(a,b),\ u_0=c_1+c_2(x-a),$ and

$$F(v)=\int_a^b fvdx+d_1v'(b)-d_2v(b)-A(u_0,v) \qquad (3.9)$$

Sufficient conditions for the existance of unique solutions of both the above problems consist of

$$0<\underline{\alpha}\le\alpha(x)\le\overline{\alpha} \quad \text{and} \quad 0<\underline{\beta}\le\beta(x)\le\overline{\beta} \qquad (3.10)$$

for positive constants $\underline{\alpha},\ \overline{\alpha},\ \underline{\beta},\ \overline{\beta}.$

**Remark 3.1:** Condition (3.10) is a sufficient condition. Actually, $\beta$ can be negative or zero. Also, $f$ can be less regular than $L^2$ function.

### 3.2 Approximate Problems and Their Solution

Taking $N\ge 3,$ let $\phi$ be the $dbN$ scaling function. Then we can approximate the above problems as

$$\left.\begin{aligned} &u_n=u_0+w_n, \quad w_n\in H_n, \\ &A(w_n,v_n)=F(v_n), \quad for\ all\ v_n\in H_n \end{aligned}\right\} \qquad (3.11)$$

where $H_n=S_n^0(a,b)$ for problem (3.1)-(3.2), and $H_n=S_n^-(a,b)$ for problem (3.1)-(3.3), where the spaces are as defined in Section 2.

**Theorem 3.1 (Error Estimate):** Let $u$ be the solution of any of the above problems and $u_n$ the solution of the approximate problem (3.11). Then for $u\in H^N(a,b),$ we have

$$\| u-u_n\|_m\le C_m h^{N-m+1}, \quad m=0,1,\ 2 \qquad (3.12)$$

where $h=2^{-n}$, $N$ is the order of the wavelet used and $C_m$ are positive constants.

**Proof:** For proof, ref. [18] can be seen.

**Remark 3.2:** The convergence of the Wavelet-Galerkin method becomes slower as the length of the interval (domain) increases.

Now, let

$$w_n=\sum_{j=1}^{p}\overline{c}_{n,j}\Phi_{n,j-S+1} \qquad (3.13)$$

produces the solution of the approximate problem (3.11) at resolution level $n\ge 0$, where $p=2^nS+S-2$. Using Galerkin method to this, we get a system of linear equations in p unknowns $\overline{c}_{n,j},\ j=1,\dots,p:$

$$A\overline{c}=F \qquad (3.14)$$

where $A$ and $F$ are the stiffness matrix and the force vector respectively. The solution of this system of equations gives rise to the approximate solution of the actual boundary value problem.

As usual, we have to apply numerical quadratures to find the matrix elements $A_{i,j}$ and the vector elements $F_i$. Indeed, we need not go through numerical differentiation as the derivatives of the basis functions are the interpolated scaling functions themselves, with a little difference. This also prevents from computing the connection coefficients for higher dimensional problems which must be done for accuracy in direct approach.

With a little exception, the matrix A here is full and the computation of all the $p^2$ elements becomes expensive with respect to time. We can overcome this drawback by applying a change of basis. Consider the functions

$$\xi_{n,k} = \Phi_{n,k-1} - \Phi_{n,k} \tag{3.15}$$

where $\{\Phi_{n,k}\}$ is the actual basis. Then it is easy to show that the set $\{\xi_{n,k}\}$ spans $H_n$ and is linearly independent and so it forms another basis for $H_n$. Note that for $0 \le k \le 2^n S - S$,

$$\text{supp}\,(\xi_{n,k}) = \text{supp}\,(\xi'_{n,k}) = \text{supp}\,(\Phi_{n,k-1}) \cup \text{supp}\,(\Phi_{n,k}).$$

So all but $S-1+N\ (=3N-2)$ of $\xi_{n,k}$ and $\xi'_{n,k}$ are locally supported in $[a,b]$ which shows that the new basis preserves the boundary conditions. Figure 3.1 shows the pictures of the modified basis functions for $S_n^0(0,1)$ and $S_n^-(0,1)$ and their derivatives for $N=3$ and $n=1$.

**Remark 3.3:** The matrix A is already sparse if $\beta = 0$ and in this case, we need not apply any change of basis.

With the modified basis at hand, the stiffness matrix A has some special structures depending on the boundary conditions of the problem. The programs must treat these structures carefully to save running time.

The system of equations (3.14) can be solved by using Gaussian elimination method or LU factorization method. Since the bilinear form $A(.,.)$ is symmetric, the matrix $A$ is symmetric and we can reduce the number of operations by using Cholesky method, if $\beta > 0$. For large resolution level $n$, the matrix $A$ is large and sparse and in that case, we can use iterative methods such as Jacobi method or method of conjugate gradient.

## IV. NUMERICAL RESULTS

Here, we perform some numerical tests to justify the quality of the method presented for the solution of fourth order boundary value problems in the last section. All the problems are solved by using $db3$, $db4$ and $db5$ scaling functions successively at resolution levels n=0, 1, 2 and 3. The solutions are compared with the exact solutions and $L^2$, $H^1$ and $H^2$ norm errors are obtained. Also we obtain finite difference solutions for comparison with the wavelet solution.

Here we do not calculate the scaling function $\phi$ explicitly which can be used from the wavelet toolbox of MATLAB with the help of the built-in-function wavefun. The function is called as

$[\phi \quad \varphi \quad x] = wavefun$ (‘dbN’, iter);

where ‘dbN’ refers to the Daubechies wavelet type, that is, db3, db4 etc, $\phi$ and $\psi$ are the associated scaling and wavelet functions respectively and iter is the number of iterations in discretizing the support $x = [0,\ S]$ to calculate $\phi$ and $\psi$ with a sample of size $2^{iter}.S$, $S = 2N-1$. At resolution level $n \ge 0$, the support $[0,\ S]$ (and so the domain $[a,\ b]$) is discretized with a sample of size $2^{n+iter}.S$. In practice, the choice of iter will affect the accuracy of the numerical quadratures and consequently the accuracy of the solution. Therefore, considerable care must be taken in choosing iter. For the evaluation of the matrix elements $A_{i,j}$ and the vector elements $F_i$, we use Simpson’s quadratures.

**International Journal of Advances in Engineering and Management (IJAEM)**
**Volume 2, Issue 4, pp: 324-331 www.ijaem.net ISSN: 2395-5252**

Test Problem 4.1

Here we solve the following problem:

$$\left.\begin{array}{l} u^{iv} + u = (1+\pi^4)\sin \pi x + 1, \quad 0 < x < 1; \\ (i)\, u(0) = 1, \quad u(1) = 1, \\ (ii)\, u''(0) = 0, \;\; u''(1) = 0 \end{array}\right\} \quad (4.1)$$

whose exact solution is given by

$$u(x) = \sin \pi x + 1. \quad (4.2)$$

The problem is solved by using $db3$, $db4$ and $db5$ wavelets at resolution levels n =0,1, 2 and 3. Also, we approximate the problem by using finite difference method with samples of size 12, 24, 48 and 96. The error for size 12 is nearly equal to that for $db3$ wavelet solution at $n = 0$. We compare the convergence rates of finite difference solution and wavelet solutions for $L^2$ error in Figure 4.1(a), for $H^1$ error in Figure 4.1(b) and for $H^2$ error in Figure 4.1(c). In Figure 4.1(c), we can see that the graphs for FD and $db3$ wavelet solutions are (almost) parallel which means that they have the same convergence rate. We know that the FDM has quadratic rate of convergence and so the db3 wavelet solution is quadratic (in $H^2$ norm), which is also predicted by Theorem 3.1. In the other cases, the convergence rates of wavelet solutions are higher than that of finite difference solution in all the three norms. The $db5$ wavelet solution at $n = 3$ is affected by roundoff errors. The linear system of equations for this problem is solved by using Gaussian elimination method.

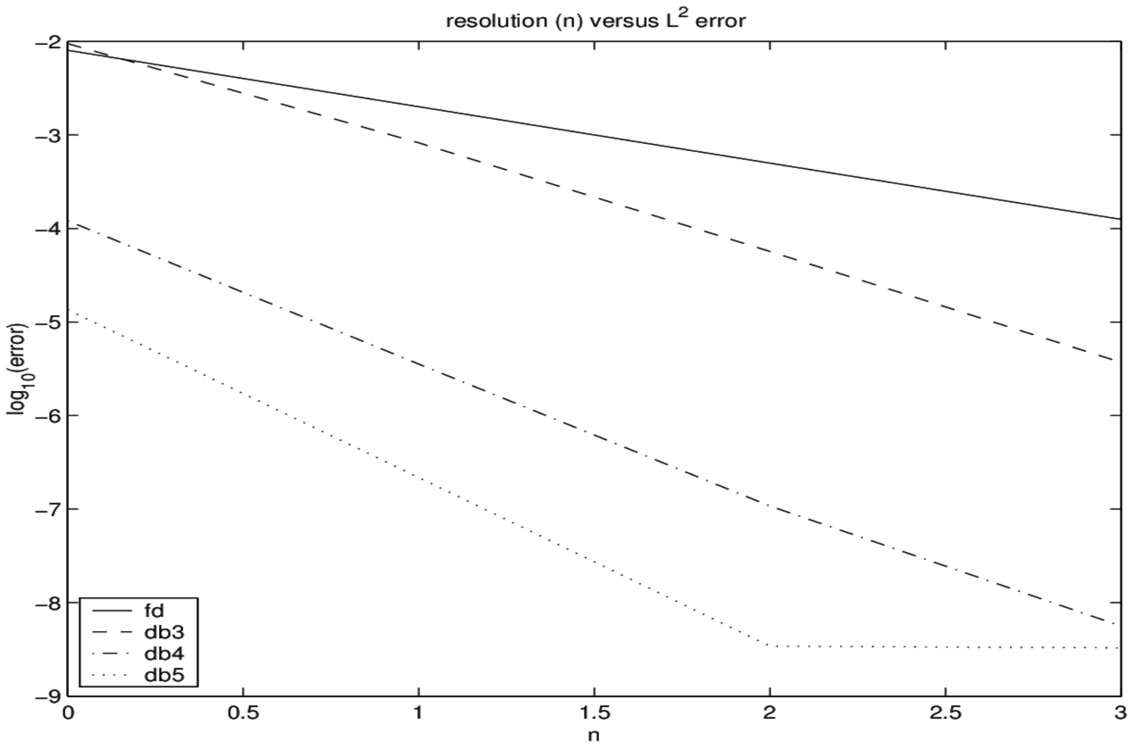


Figure 4.1(a): Test Problem 4.1. Decay in $L^2$ error with increasing resolution

International Journal of Advances in Engineering and Management (IJAEM)
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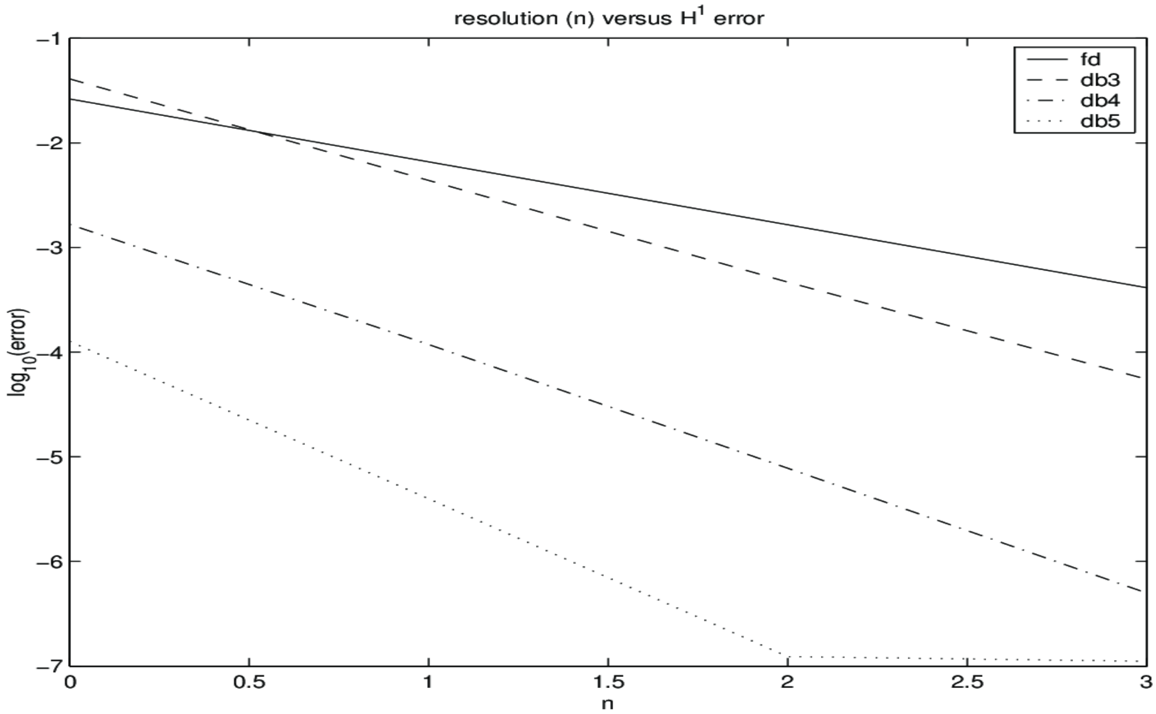


Figure 4.1(b): Test Problem 4.1. Decay in $H^1$ error with increasing resolution

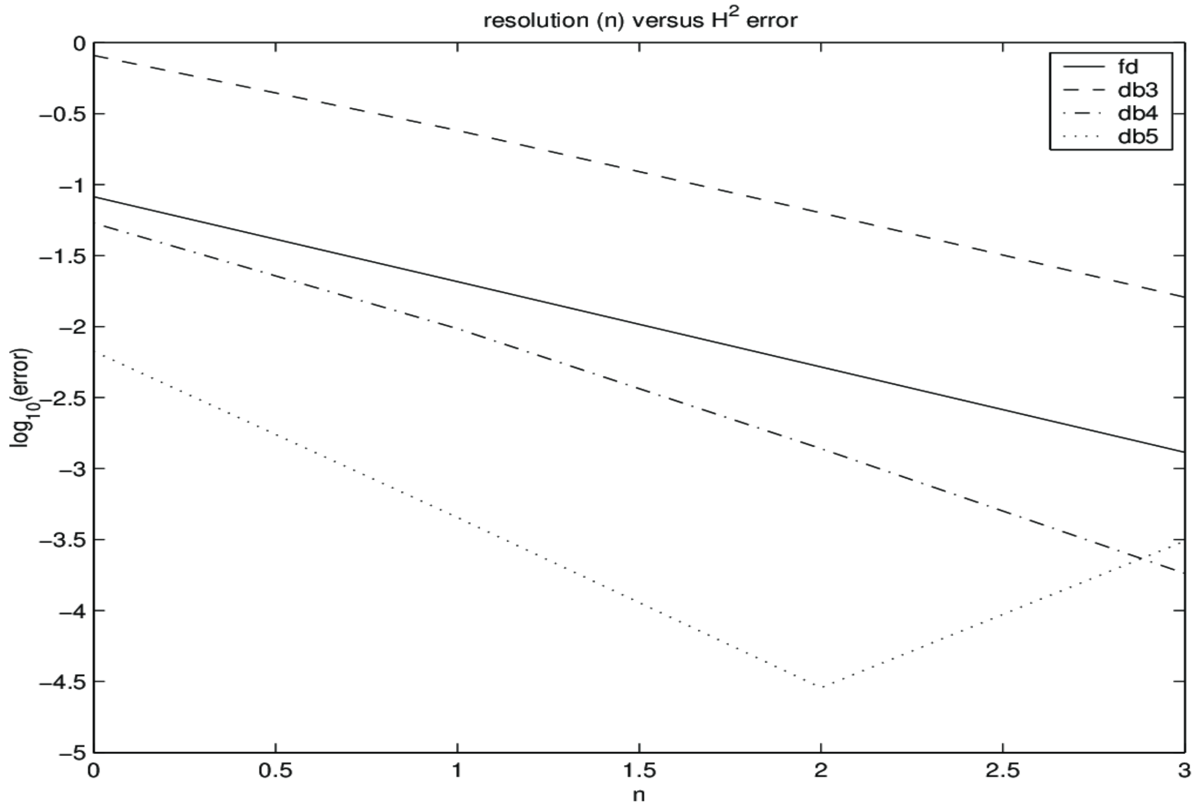


Figure 4.1(c): Test Problem 4.1. Decay in $H^2$ error with increasing resolution

Test Problem 4.2

Here we solve the following problem:

$$\left.\begin{array}{l} u^{iv} + u = (1+\pi^4)\sin \pi x + 1, \quad 1 < x < 2; \\ (i)\, u(1) = 1, \quad u'(1) = -\pi, \\ (ii)\, u''(2) = 0, \quad u'''(2) = -\pi^3 \end{array}\right\} \tag{4.3}$$

whose exact solution is given by

$$u(x) = \sin \pi x + 1. \tag{4.4}$$

As for the last problem, this problem is also solved by using $db3, db4$ and $db5$ wavelets at resolution levels n = 0, 1, 2 and 3. Also, we approximate the problem by using finite difference method with samples of size 18, 36, 72 and 144. The $L^2$ error for size 18 is nearly equal to that for $db3$ wavelet solution at n = 0. We compare the convergence rates of finite difference solution and wavelet solutions for $L^2$ error in Figure 4.2(a), for $H^1$ error in Figure 4.2(b) and for $H^2$ error in Figure 4.2(c). As in the last problem, Figure 4.2(c) verifies that $db3$ wavelet solution is quadratic in $H^2$ norm, which is also predicted by Theorem 3.1. All the three figures justify the supremacy of the wavelet solutions. The $db5$ wavelet solution at $n = 3$ is affected by roundoff errors. For this problem also, the system of linear equations is solved by using Gaussian elimination method.

## V. CONCLUSION

In this paper, we have been making attempts to construct suitable basis functions, for Galerkin solutions of one dimensional elliptic problems of fourth order with various types of boundary conditions, from Daubechies scaling functions. A similar procedure for second order problems has been discussed in the authors' paper [6]. Instead of scaling functions, wavelet functions can also be employed. The comparison of the solutions with FDM indicates that for these types of problems, wavelet method is superior over the other classical methods. The methods and techniques described here can be generalized to multidimensional problems by combining with fictitious domain methods as in Wells and Zhou [17] or in Glowinski et. al. [8].

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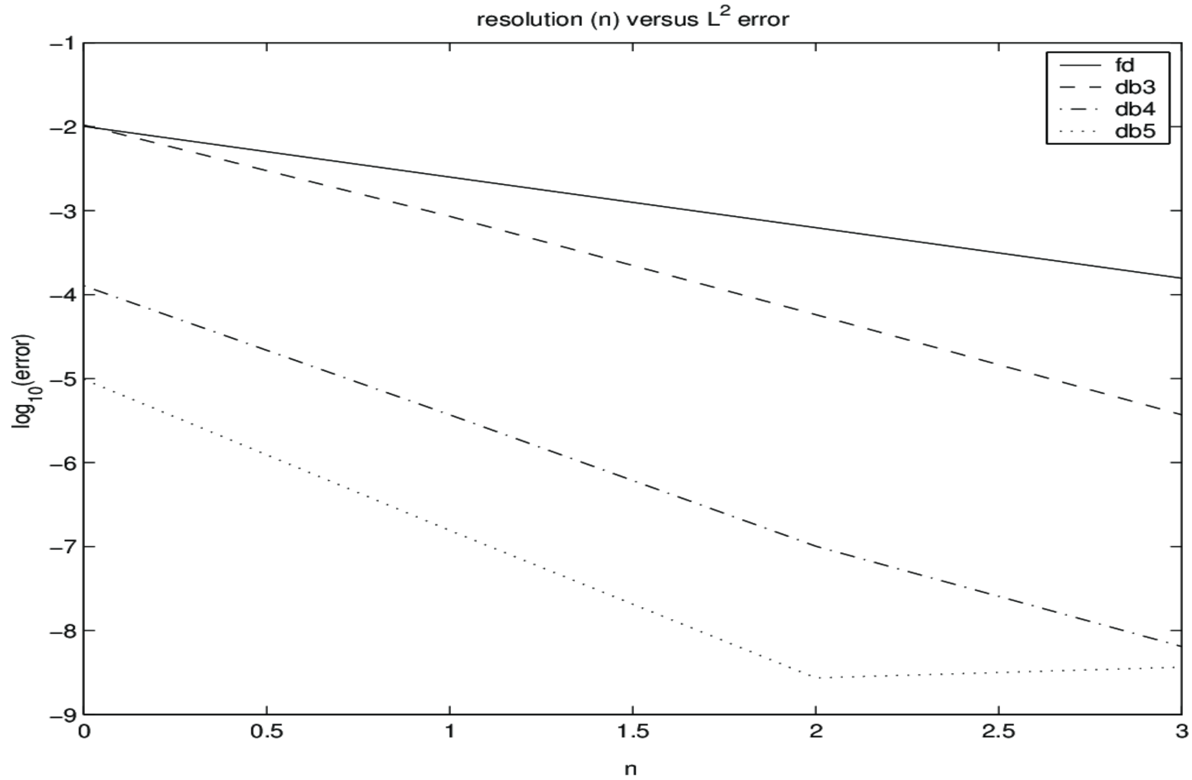


Figure 4.2(a): Test Problem 4.2. Decay in $L^2$ error with increasing resolution

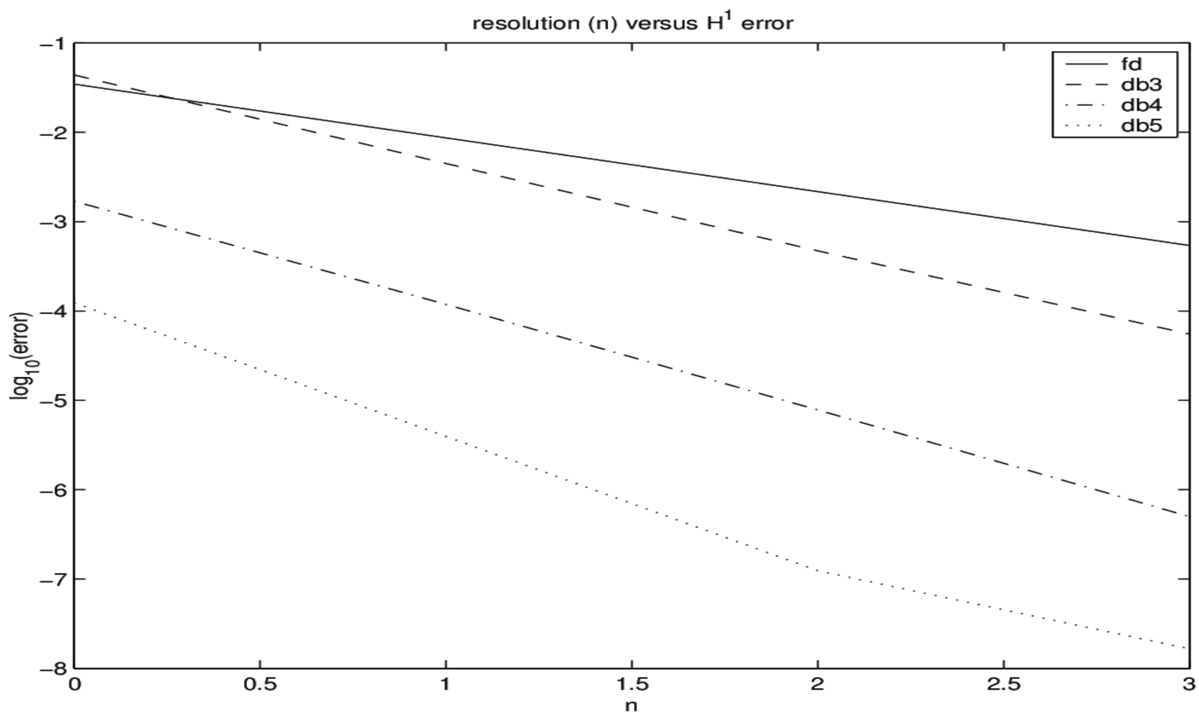


Figure 4.2(b): Test Problem 4.2. Decay in $H^1$ error with increasing resolution

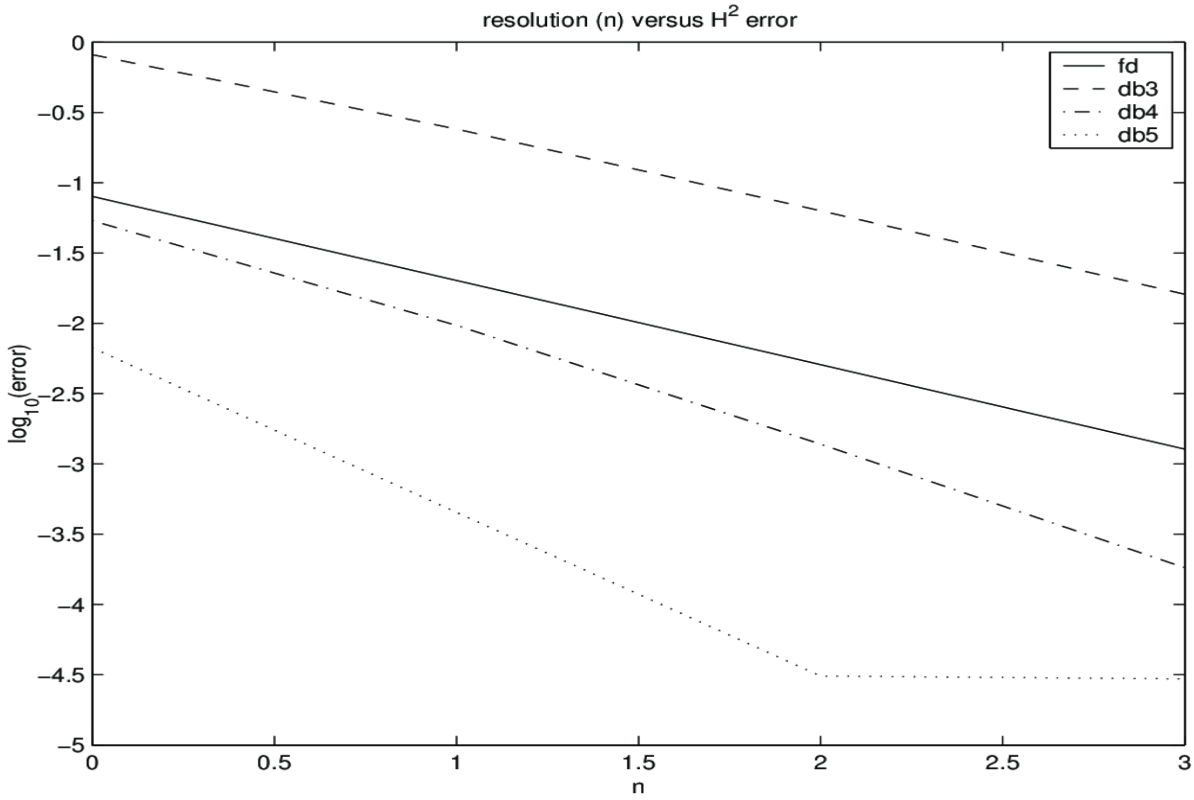


Figure 4.2(c): Test Problem 4.2. Decay in $H^2$ error with increasing resolution

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